

# LOOP SPACES AND REPRESENTATIONS

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**ABSTRACT.** We introduce loop spaces (in the sense of derived algebraic geometry) into the representation theory of reductive groups. In particular, we apply the theory developed in [BN2] to flag varieties, and obtain new insights into fundamental categories in representation theory. First, we show that one can recover finite Hecke categories (realized by  $\mathcal{D}$ -modules on flag varieties) from affine Hecke categories (realized by coherent sheaves on Steinberg varieties) via  $S^1$ -equivariant localization. Similarly, one can recover  $\mathcal{D}$ -modules on the nilpotent cone from coherent sheaves on the commuting variety. We also show that the categorical Langlands parameters for real groups studied by Adams-Barbasch-Vogan [ABV] and Soergel [So] arise naturally from the study of loop spaces of flag varieties and their Jordan decomposition (or in an alternative formulation, from the study of local systems on a Möbius strip). This provides a unifying framework that overcomes a discomforting aspect of the traditional approach to the Langlands parameters, namely their evidently strange behavior with respect to changes in infinitesimal character.

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## 1. INTRODUCTION

This is the second paper in a two paper series<sup>1</sup>. In the first paper [BN2], we studied loop spaces in derived algebraic geometry, in particular the relation between sheaves on loop spaces and connections on the base. In this paper, we apply this theory to flag varieties of reductive groups and find consequences for the representation theory of Hecke algebras and Lie groups.

<sup>1</sup>It is a strengthened version of the second half of the preprint [BN07].

It is well known that connections in the form of  $\mathcal{D}$ -modules play a central role in geometric representation theory. The theory of the first part [BN2] establishes the intimate relation between  $\mathcal{D}$ -modules and the geometry of so called “small loops”, or more precisely, loops in the formal neighborhood of constant loops. Thus with Beilinson-Bernstein localization in mind, it is not surprising that small loops provide an alternative language to discuss constructions of representations. The striking feature of this second part is that all loops arise naturally in the dual *Langlands parametrization* of representations. Essential to this realization is the notion, motivated by rational homotopy theory, of *unipotent loops* introduced in [BN2]. In fact, we describe a pattern of Jordan decomposition for loop spaces which precisely accounts for the seemingly erratic behavior of the Langlands parameters as functions of infinitesimal character. Thus we will see that the notion of loop space organizes much of the seeming cacophany of spaces parametrizing representations of Lie groups.

We summarize our main results immediately below. We have organized the review into three parts: the first related to Hecke categories, the second to representations of real groups, and the third to the nilpotent cone and commuting variety.

In Section 2, we briefly review the results of [BN2] on loop spaces in derived algebraic geometry. This will provide the bridge from quasicoherent sheaves to  $\mathcal{D}$ -modules.

In Section 3, we present our results on Steinberg varieties and Hecke categories.

In Section 4, we present our results on Langlands parameters for real groups.

*Conventions.* We work throughout over a rational algebra  $k$ . Our main results are equivalences of pre-triangulated  $k$ -linear differential graded (dg) categories. One can view such dg categories as objects of the  $\infty$ -category of stable  $k$ -linear  $\infty$ -categories. Without any further comment, all categorical constructions and assertions are assumed to take place therein.

By a  $\mathcal{D}$ -module on a smooth stack  $X$  we will always mean a  $\mathcal{D}$ -module that admits a reasonable filtration, i.e., such that the resulting graded Rees module is complete. We will call such  $\mathcal{D}$ -modules *reasonable*. Note that any coherent  $\mathcal{D}$ -module is reasonable (thanks to good filtrations). The reason for this restriction is that reasonable  $\mathcal{D}$ -modules are identified with de Rham modules via Koszul duality. See Section 2.0.3 for a discussion.

*Notation.* For easy reference, here is a summary of notation used throughout the paper. Let  $G$  be a connected reductive complex algebraic group with Lie algebra  $\mathfrak{g}$ . Let  $\mathcal{B}$  be the flag variety of  $G$  parameterizing Borel subgroups  $B \subset G$ . For each  $B \in \mathcal{B}$ , we have the Cartan quotient  $H = B/U$  where  $U \subset B$  is the unipotent radical. The natural conjugation  $G$ -action on  $\mathcal{B}$  canonically identifies the Cartan quotients for different  $B$ , and so it is justified to call  $H$  the universal Cartan. It is also convenient to choose a maximal torus  $T \subset B$  with normalizer  $N(T) \subset G$ , and Weyl group  $W = N(T)/T$ .

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**1.1. Hecke categories.** Our first main result provides a direct relation between two of the fundamental categories in representation theory.

**Definition 1.1.** Fix a complex reductive group  $G$  with Borel subgroup  $B \subset G$ , and let  $G^u \subset G$  denote the formal neighborhood of the unipotent elements.

The *finite Hecke category* is the dg derived category  $\mathcal{H}_G = \mathcal{D}(B \backslash G/B)$  of  $B$ -equivariant  $\mathcal{D}$ -modules on the flag variety  $\mathcal{B} \simeq G/B$ .

The *affine Hecke category* is the dg derived category  $\mathcal{H}_G^{\text{aff}} = \text{QCoh}(\mathcal{S}t^u/G)$  of  $G$ -equivariant quasicoherent sheaves on the (unipotent) Steinberg variety

$$\mathcal{S}t^u = \{(g, B_1, B_2) \in G^u \times \mathcal{B} \times \mathcal{B} \mid g \in B_1 \cap B_2\}.$$

*Remark 1.2.* Dilation of unipotent elements equips  $\mathcal{S}t^u$  and hence the affine Hecke category  $\mathcal{H}_G^{\text{aff}}$  with a canonical  $\mathbb{G}_m$ -action. (Our convention is that the structure sheaf  $\mathcal{O}_{\mathcal{S}t^u}$  has non-positive  $\mathbb{G}_m$ -weights.) This extra symmetry, familiar in representation theory, will play an immediate key role in what follows.

The starting point of this work is the following observation: one can realize  $\mathcal{S}t^u/G$  as a kind of loop space in the setting of derived algebraic geometry. Hence it carries a corresponding action of the circle  $S^1$ , which can be expressed completely in terms of the cochains

$$\mathcal{O}(S^1) = C^*(S^1, k) \simeq H^*(S^1, k) = k[\eta]/(\eta^2), \quad |\eta| = 1.$$

More precisely,  $\mathcal{S}t^u/G$  is a *unipotent loop space*, a notion introduced in [BN2] (with motivation from rational homotopy theory) and reviewed in Section 2. As a result the circle action factors through an action of the affinization

$$\text{Aff}(S^1) = \text{Spec } \mathcal{O}(S^1).$$

Moreover, the dilation  $\mathbb{G}_m$ -action on  $\mathcal{S}t^u/G$  is naturally induced by the formality of  $\mathcal{O}(S^1)$ . It follows (as for any unipotent loop space) that the  $S^1$ -action is compatible with the  $\mathbb{G}_m$ -action, and they combine into the action of the single semi-direct product group

$$\mathbb{S} = \text{Aff}(S^1) \rtimes \mathbb{G}_m.$$

Recall that any  $S^1$ -equivariant category  $\mathcal{C}$ , and hence any  $\mathbb{S}$ -equivariant category, is naturally linear over the global functions

$$\mathcal{O}(BS^1) \simeq H^*(BS^1, k) \simeq k[h], \quad |h| = 2.$$

We write  $\mathcal{C}_{loc}$  for the localization of  $\mathcal{C}$  where we invert the action of the generator  $h$ . Since the cohomological degree of  $h$  is two, the morphisms of  $\mathcal{C}_{loc}$  are naturally  $\mathbf{Z}/2\mathbf{Z}$ -graded rather than  $\mathbf{Z}$ -graded. But if  $\mathcal{C}$  is any  $\mathbb{S}$ -equivariant category, then thanks to the additional  $\mathbb{G}_m$ -weight grading, the morphisms of  $\mathcal{C}_{loc}$  are again naturally  $\mathbf{Z}$ -graded.

Now we arrive at our first main result. Let  $(\mathcal{H}_G^{\text{aff}})^{\mathbb{S}^-}$  denote the full subcategory of  $\mathbb{S}$ -invariants of the affine Hecke category with bounded above  $\mathbb{G}_m$ -weight.

**Theorem 1.3.** *There is a canonical equivalence*

$$\mathcal{H}_G \simeq (\mathcal{H}_G^{\text{aff}})_{loc}^{\mathbb{S}^-}$$

*between the finite Hecke category and the localized  $\mathbb{S}$ -invariants with bounded above weight of the affine Hecke category.*

*Remark 1.4.* Although we do not discuss monoidal structures in this paper, one can check that the above equivalence is naturally monoidal with respect to convolution.

The proof of Theorem 1.3 involves two parts: first, making precise the relation between  $\mathcal{S}t^u/G$  and the loop space of  $B \backslash G/B$ , and second, applying general results on loop spaces and connections from [BN2].

A striking aspect of the theorem is its mixture of quasicoherent sheaves and  $\mathcal{D}$ -modules. We were led to it via the duality of local tamely ramified Hecke operators in the Geometric Langlands program. In what immediately follows, we informally explain this motivating picture.

Let  $G^\vee$  denote the Langlands dual group, let  $LG^\vee$  be its loop group of maps  $\text{Spec } C((t)) \rightarrow G^\vee$ , and let  $I \subset LG^\vee$  be an Iwahori subgroup. The traditional definition of the affine Hecke

category is the dg derived category  $D_u(I \backslash LG^\vee / I)$  of perverse sheaves on  $LG^\vee$  which are Iwahori-bimonodromic with unipotent monodromy. A deep theorem of Bezrukavnikov [Be] (with groundbreaking applications to Lusztig's vision of the representation theory of Lie algebras in characteristic  $p$ , quantum groups, and affine algebras) lifts the Kazhdan-Lusztig realizations of affine Hecke algebras [KL] (or see [CG] for an exposition) to the level of derived categories

$$\mathrm{Coh}_G(\mathcal{S}t^u) \simeq D_u(I \backslash LG^\vee / I).$$

Our discovery of Theorem 1.3 was inspired by a more evident parallel picture on the other side of the above equivalence. Namely, by its very definition  $LG^\vee$  is a loop space, and loop rotation equips  $D_u(I \backslash LG^\vee / I)$  with a natural  $S^1$ -action. The  $S^1$ -fixed points in  $LG^\vee$  are the constant loops  $G^\vee$ , and one can show that the localized  $S^1$ -invariants in  $D_u(I \backslash LG^\vee / I)$  form (a periodic version of) the  $B$ -bimonodromic finite Hecke category  $D_u(B^\vee \backslash G^\vee / B^\vee)$ . This is a relatively straightforward application of equivariant localization in topology, for example using the differential graded techniques developed by Goresky-Kottwitz-MacPherson [GKM].

Soergel [So] interpreted the Koszul duality theorem of Beilinson-Ginzburg-Soergel [BGS] in the context of Langlands duality, identifying (graded or 2-periodic versions of) the finite Hecke categories  $D_u(B^\vee \backslash G^\vee / B^\vee)$  and  $\mathcal{D}(B \backslash G / B)$ . It is interesting to note that as a consequence of the above perspective, we recover the Langlands duality of finite Hecke categories from Bezrukavnikov's equivalence of affine Hecke categories by equivariant localization. In particular, this fixes a canonical monoidal form of this duality. (For a direct verification of monoidal aspects of this equivalence, see Bezrukavnikov-Yun [BY].)

**1.2. Langlands parameters.** Our second main result gives a uniform geometric description of the categorical Langlands parameters for representations of real groups.

Let  $G^\vee$  denote the Langlands dual group, and let  $\theta$  be a quasi-split conjugation of  $G^\vee$ . Adams-Barbasch-Vogan [ABV] and Soergel [So] introduce and investigate categories of equivariant  $\mathcal{D}$ -modules which parametrize Harish Chandra modules for real forms of  $G^\vee$  in the inner class of  $\theta$ . For a fixed infinitesimal character, the  $\mathcal{D}$ -modules live on flag varieties constructed out of  $G$ , an involution  $\eta$  corresponding to  $\theta$ , and the infinitesimal character. On the level of Grothendieck groups, the parametrization is a form of Vogan's character duality [V], while on the level of derived categories, it is a Koszul duality conjecture due to Soergel [So].

We have aimed to understand this circle of ideas through the lens of topological field theory. For example, in the paper [BN1], we construct a partial 3d TFT which governs the above objects and has a natural Langlands duality. Our aim here is to deduce that the parameter categories of [ABV] and [So] naturally arise from the 4d Geometric Langlands TFT. We will not go into the details of TFT, but rather highlight some of the consequences of this viewpoint, in particular a new connection of the parameter categories to affine Hecke categories and a new uniform description of them with respect to regular infinitesimal character.

**Definition 1.5.** For an involution  $\eta$  of  $G$ , the *Langlands parameter variety*  $\mathcal{L}$  is defined to be

$$\mathcal{L} = \{(g, B) \in G \times \mathcal{B} \mid g\eta(g) \in B\}.$$

For an element  $\alpha$  of the universal Cartan  $H$ , we define the *monodromic Langlands parameter variety*  $\mathcal{L}_\alpha \subset \mathcal{L}$  to be the subscheme of pairs  $(g, B)$  such that the semisimple part of  $g\eta(g)$ , with respect to the flag  $B$ , is in the formal neighborhood of  $\alpha$ .

The group  $G$  acts on  $\mathcal{L}$  by twisted conjugation, and one can realize  $\mathcal{L}/G$  as a kind of loop space. Thanks to a Jordan decomposition pattern for loops, each of the substacks  $\mathcal{L}_\alpha/G$  has a natural loop space interpretation resulting in  $\mathbb{S}$ -actions on each  $\mathcal{L}_\alpha/G$ .

*Remark 1.6.* In fact, one can realize  $\mathcal{L}/G$  as a moduli of local systems on the Möbius strip with a flag along the boundary. From this viewpoint, the  $S^1$ -action on  $\mathcal{L}/G$  comes from rotating the Möbius strip. It is this picture, originating in topological field theory, which motivated the definition of  $\mathcal{L}/G$ .

We will consider the dg derived category  $\mathrm{QCoh}(\mathcal{L}_\alpha/G)$  of  $G$ -equivariant quasicoherent sheaves on the monodromic Langlands parameter variety  $\mathcal{L}_\alpha$ .

*Remark 1.7.* We do not investigate this structure here, but it is noteworthy that  $\mathrm{QCoh}(\mathcal{L}_\alpha/G)$  is naturally a module category for the appropriate monodromic affine Hecke category.

Our second main result is the following theorem. We state it more precisely in Theorem 1.11 immediately below after developing further notation.

**Theorem 1.8** (Informal version). *For any regular  $\lambda \in \mathfrak{h}$ , the Langlands parameter category for infinitesimal character  $[\lambda] \in \mathfrak{h}/W$  is canonically equivalent to the localized  $\mathbb{S}$ -invariants with bounded above weight  $\mathrm{QCoh}(\mathcal{L}_\alpha/G)_{loc}^{\mathbb{S}-}$ , where  $\alpha = \exp(\lambda) \in H$ .*

*Remark 1.9.* Rotating the Möbius strip equips  $\mathcal{L}/G$ , and hence  $\mathcal{L}_\alpha/G$ , with a natural  $S^1$ -action. But in general, there is no evident compatible  $\mathbb{G}_m$ -action. We will explain that in fact  $\mathcal{L}_\alpha/G$  is itself a unipotent loop space and thus has a canonical  $\mathbb{S}$ -action. The corresponding  $S^1$ -action coincides up to a central automorphism with that obtained by rotating the Möbius strip.

*Remark 1.10.* In the case of a complex group (considered as a real form of its complexification), the theorem is a version (with parameter  $\alpha \in H \times H$ ) of Theorem 1.3 (to which it reduces when  $\alpha = (e, e)$ ) which is spelled out in detail in Section 3.4.

The precise version of Theorem 1.8 proved in the paper (see Corollaries 4.13 and [?]) is self-contained and makes no reference to the theory of [ABV] and [So]. It gives a direct and concrete description of  $\mathrm{QCoh}(\mathcal{L}_\alpha/G)_{loc}^{\mathbb{S}-}$  in terms of equivariant  $\mathcal{D}$ -modules on flag varieties. One can then check that our description coincides with the corresponding Langlands parameter category of [ABV] and [So] for regular infinitesimal character.

Now in order to state a more precise version of Theorem 1.8, we include here a slightly expanded review of the Langlands parameter categories for representations of real groups. Recall that  $G^\vee$  denotes the Langlands dual group, and  $\theta$  a quasi-split conjugation of  $G^\vee$  so that  $\eta$  is a corresponding involution of  $G$ .

Associated to  $\theta$  is a finite collection (possibly with multiplicities)  $\Theta(\theta)$  of conjugations of  $G^\vee$  all in the same inner class as  $\theta$ . For each  $\tau \in \Theta(\theta)$ , we write  $G_{\mathbf{R},\tau}^\vee \subset G^\vee$  for the corresponding real form. For each  $[\lambda] \in \mathfrak{h}/W \simeq (\mathfrak{h}^\vee)^*/W$ , we write  $\mathcal{HC}_{\tau,[\lambda]}$  for the dg derived category of Harish Chandra modules for the real form  $G_{\mathbf{R},\tau}^\vee$  with (pro-completed) generalized infinitesimal character  $[\lambda]$ . For simplicity, we will restrict our attention to the case when  $[\lambda]$  is regular.

Fix a semisimple lift  $\lambda \in \mathfrak{g}$  of the infinitesimal character  $[\lambda]$ , and let  $\alpha \in G$  denote the element  $\exp(\lambda)$ . Let  $G_\alpha \subset G$  be the reductive subgroup that centralizes  $\alpha$ , and let  $\mathcal{B}_\alpha = G_\alpha/B_\alpha$  be its flag variety. Consider the finite set of twisted conjugacy classes

$$\Sigma(\eta, \alpha) = \{\sigma \in G \mid \sigma\eta(\sigma) = \alpha\}/G.$$

Each  $\sigma \in \Sigma(\eta, \alpha)$  defines an involution of  $G_\alpha$ , and we write  $K_{\alpha,\sigma} \subset G_\alpha$  for the corresponding symmetric subgroup.

Soergel [So] conjectures that there should be a Koszul duality between categories of Harish Chandra modules and Langlands parameters

$$\bigoplus_{\tau \in \Theta(\theta)} \mathcal{HC}_{\tau,[\lambda]} \overset{?}{\longleftrightarrow} \bigoplus_{\sigma \in \Sigma(\eta,\alpha)} \mathcal{D}(K_{\alpha,\sigma} \backslash G_\alpha/B_\alpha)$$

respecting Hecke symmetries. Soergel establishes this conjecture in the case of tori,  $SL_2$  and most importantly, for complex groups  $G^\vee$  (considered as real forms of their complexifications). In the complex case, the conjecture (for  $\alpha = (e, e)$ ) reduces to the Langlands duality for finite Hecke categories described above.

Soergel's conjecture is a categorical form of the results of Adams, Barbasch and Vogan [ABV], who found an interpretation of the Langlands parametrization of admissible representations of real groups in terms of equivariant sheaves. These authors deduce the above conjecture on the level of Grothendieck groups from Vogan's character duality [V], and combine it with the microlocal geometry of the cotangent bundles  $T^*(K_{\alpha,\sigma} \backslash G_\alpha / B_\alpha)$  to study Arthur's conjectures. As mentioned in [ABV], it is important to find a way to fit together the spaces  $K_{\alpha,\sigma} \backslash G_\alpha / B_\alpha$ , for varying  $\alpha$ . In particular, this is necessary if one hopes to have a uniform picture for representations with different infinitesimal characters.

One of the outcomes of this paper is a solution to this problem in the form of the Langlands parameter variety  $\mathcal{L}$ . The crucial change of perspective is that loop spaces rather than cotangent bundles are a more natural classical format for encoding the quantum geometry. The key insight is an identification

$$\mathcal{L}_\alpha / G \simeq \coprod_{\sigma \in \Sigma(\eta, \alpha)} \mathcal{L}^u(K_{\alpha,\sigma} \backslash G_\alpha / B_\alpha)$$

proved in Corollary 4.13. In particular, as a unipotent loop space,  $\mathcal{L}_\alpha / G$  comes equipped with a natural  $\mathbb{S}$ -action. Now with the above preparations behind us, we can state a more precise version of Theorem 1.8.

**Theorem 1.11.** *For any  $\alpha \in H$ , there is a canonical equivalence*

$$\mathrm{QCoh}(\mathcal{L}_\alpha / G)_{loc}^{\mathbb{S}_-} \simeq \bigoplus_{\sigma \in \Sigma(\eta, \alpha)} \mathcal{D}(K_{\alpha,\sigma} \backslash G_\alpha / B_\alpha)$$

*between the localized  $\mathbb{S}$ -invariants with bounded above weight in  $\mathrm{QCoh}(\mathcal{L}_\alpha / G)$  and the Langlands parameter category.*

The proof of Theorem 1.11 involves a detailed study of the interaction of the involution  $\eta$  with the equivalence of Theorem 1.3. A remarkable aspect of this argument is the appearance of the complicated combinatorics of parameters for Harish Chandra modules from nothing more than a formal construction involving the well understood combinatorics of Weyl groups.

**1.3. Nilpotent cone and commuting variety.** We conclude with a final simple illustration of the general theory of [BN2]. We will realize  $\mathcal{D}$ -modules on the nilpotent cone in terms of quasicoherent sheaves on the commuting variety.

We first consider  $\mathcal{D}$ -modules on the adjoint quotient  $G/G$  from the perspective of loop spaces. As recalled in Section 2 below, we are thus led to consider the double loop space

$$\mathcal{L}(G/G) = \mathcal{L}(\mathcal{L}BG) \simeq \mathrm{Map}(T^2, BG) \simeq \mathrm{Loc}_G(T^2),$$

the derived stack of maps from the two-torus  $T^2$  into  $BG$ , or equivalently, of  $G$ -local systems on  $T^2$ . By writing  $T^2$  as a wedge of circles with a disk attached, we can describe  $\mathcal{L}(G/G)$  as the derived stack of pairs of commuting elements up to conjugation

$$\mathcal{L}(G/G) \simeq \{(g_1, g_2) \in G \times G \mid g_1 g_2 = g_2 g_1\} / G.$$

In other words,  $\mathcal{L}(G/G)$  is the derived fiber product modulo conjugation

$$\mathcal{L}(G/G) \simeq ((G \times G) \times_G \{e\}) / G$$

where the first map is the commutator  $(g_1, g_2) \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$ , and the second is the inclusion of the identity  $e \in G$ .

In order to describe  $\mathcal{D}$ -modules on  $G/G$ , we need in fact only consider unipotent loops in  $G/G$ , which are easily seen to be

$$\mathcal{L}^u(G/G) \simeq \{(g_1, g_2) \in G \times G^u \mid g_1 g_2 = g_2 g_1\} / G.$$

where as usual  $G^u \subset G$  denotes the formal neighborhood of unipotent elements. As an immediate consequence of results of [BN2] recalled in Section 2 below, we conclude that  $\mathcal{D}(G/G)$  is equivalent to the localized  $\mathbb{S}$ -invariants with bounded above weight in the dg category of quasicoherent sheaves on  $\mathcal{L}^u(G/G)$ .

We find a more symmetric situation if we require the initial loop represented by  $g_1 \in G$  to be unipotent as well.

**Definition 1.12.** The *nilpotent cone* is the subvariety  $\mathcal{N} \subset \mathfrak{g}$  of nilpotent elements.

The *unipotent commuting variety* is the derived scheme

$$\mathcal{C}^u = \{(g_1, g_2) \in G^u \times G^u \mid g_1 g_2 = g_2 g_1\}.$$

The double unipotent loop space  $\mathcal{C}^u/G = \mathcal{L}^u(\mathcal{L}^u(BG))$  naturally classifies  $G$ -local systems on the two-torus  $T^2$  with unipotent monodromies. Observe that  $\mathcal{C}^u/G$  carries two commuting  $\mathbb{G}_m$ -actions given by rescaling the two unipotent group elements. Moreover, its realization as double loops (or as local systems) makes apparent two commuting  $S^1$ -actions. Altogether, we have an action of the product  $\mathbb{S} \times \mathbb{S}$ . In the following theorem (whose proof is an easy exercise in the techniques of [BN2] and the present paper), we will restrict our attention to the second factor of  $\mathbb{S}$ .

**Theorem 1.13.** *There is a canonical equivalence*

$$\mathcal{D}(\mathcal{N}/G) \simeq \mathrm{QCoh}(\mathcal{C}^u/G)_{loc}^{\mathbb{S}_-}$$

*between equivariant  $\mathcal{D}$ -modules on the nilpotent cone and the localized  $\mathbb{S}$ -invariants with weight bounded above in equivariant quasicoherent sheaves on the commuting variety.*

*Remark 1.14.* It would be interesting to consider the fully invariant version  $\mathrm{QCoh}(\mathcal{C}^u/G)^{\mathbb{S}_- \times \mathbb{S}_-}$  and its possible localizations. Any construction symmetric with respect to the two circles in the two-torus will carry a canonical  $SL_2(\mathbf{Z})$ -action.

Theorem 1.13 connects the category  $\mathcal{D}(\mathcal{N}/G)$ , which is at the heart of Springer theory, with another category of great current interest, see in particular the recent work of Schiffmann and Vasserot [SV1, SV2] and Ginzburg [G]. The commuting variety (in the case of  $GL_n$ ) is closely related to the Hilbert scheme of points in  $\mathbf{C}^n$ , and its Grothendieck group of  $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant coherent sheaves is linked to the theory of Macdonald polynomials and double affine Hecke algebras<sup>2</sup>. In particular, the interpretation via local systems on the torus identifies this  $K$ -group (via the geometric Langlands conjecture) with the elliptic Hall algebra and the theory of Eisenstein series.

## 2. LOOP SPACES AND FLAT CONNECTIONS

We review here some of the relevant results of [BN2]. It is written in the language of derived algebraic geometry over a fixed  $\mathbf{Q}$ -algebra  $k$ , and in this paper, we will always work over  $\mathbf{C}$ . As it turns out, the geometric objects which arise in this paper (with the exception of the commuting variety) are ordinary Artin stacks (in fact, quotients of quasi-projective varieties by reductive groups), but we will make substantial use of results which depend on the flexible context of derived stacks.

<sup>2</sup>While the authors cited above consider commuting elements in the Lie algebra rather than in the formal neighborhood  $G^u$  of the unipotent cone, their  $\mathbb{G}_m \times \mathbb{G}_m$ -equivariant  $K$ -groups are equivalent.

In broad outline, to arrive at derived algebraic geometry from classical algebraic geometry, one takes the following steps. First, one replaces commutative  $k$ -algebras by *derived  $k$ -algebras*, namely simplicial (or equivalently, for  $k$  a  $\mathbf{Q}$ -algebra, connective differential graded) commutative  $k$ -algebras. This allows for derived intersections by replacing the tensor product of rings with its derived functor. Second, one considers functors of points on derived rings that take values not only in sets but in simplicial sets or equivalently topological spaces. This allows for derived quotients by keeping track of gluings in the enriched theory of spaces.

Roughly speaking, a *derived stack* is a functor from derived rings to topological spaces satisfying an sheaf axiom with respect to the étale topology on derived rings. Examples include all of the schemes of classical algebraic geometry, stacks of modern algebraic geometry, along with topological spaces in the form of locally constant stacks. Any derived stack has an “underlying” underived stack, obtained by restricting its functor of points to ordinary rings. In fact, derived stacks can be viewed as formal thickenings of underived stacks, just as supermanifolds are infinitesimal thickenings of manifolds.

One of the complicated aspects of the theory is that the domain of derived rings, target of topological spaces, and functors of points themselves must be treated with the correct enriched homotopical understanding. We recommend Toën’s excellent survey [To] (as well as [BN2]) for more details and references. It was our introduction to many of the notions of derived algebraic geometry, in particular derived loop spaces.

**2.0.1. Three kinds of loops.** We consider the circle  $S^1 = B\mathbf{Z} = K(\mathbf{Z}, 1)$  as a (locally constant) stack. Since we work in characteristic zero, its algebra of cochains is formal

$$\mathcal{O}(S^1) = C^*(S^1, k) \simeq H^*(S^1, k) = k[\eta]/(\eta^2), \quad |\eta| = 1.$$

In fact,  $\mathcal{O}(S^1)$  is the free symmetric algebra (in the graded sense) on a single generator of degree one. Thus its affinization (the spectrum of its algebra of cochains) is an odd version of the affine line, namely the classifying stack of the additive group

$$\mathrm{Aff}(S^1) \simeq B\mathbb{G}_a \simeq K(\mathbb{G}_a, 1) \simeq \mathbf{A}^1[1].$$

This is the stack that assigns to a  $k$ -algebra  $R$  the groupoid  $BR$ , or equivalently the Eilenberg-MacLane space  $K(R, 1)$ , where we consider  $R$  as an additive group. We will prefer the notation  $B\mathbb{G}_a$  to emphasize the group structure on  $\mathrm{Aff}(S^1)$ .

**Definition 2.1.** The *loop space* of a derived stack  $X$  is the derived mapping stack

$$\mathcal{L}X = \mathrm{Map}_{DSt}(S^1, X).$$

The *unipotent loop space* of  $X$  is the derived mapping stack

$$\mathcal{L}^u X = \mathrm{Map}_{DSt}(\mathrm{Aff}(S^1), X).$$

The *formal loop space* of  $X$  is the formal completion of  $\mathcal{L}X$  along the constant loops

$$\widehat{\mathcal{L}}X = \widehat{\mathcal{L}X}_X.$$

Pulling back along the affinization homomorphism  $S^1 \rightarrow \mathrm{Aff}(S^1)$  defines a morphism  $\mathcal{L}^u X \rightarrow \mathcal{L}X$ . This is an equivalence when  $X$  is a derived scheme (quasi-compact with affine diagonal). More generally, when  $X$  is a geometric stack (Artin with affine diagonal), formal loops are unipotent: the morphism factors

$$\widehat{\mathcal{L}}X \rightarrow \mathcal{L}^u X \rightarrow \mathcal{L}X.$$



**2.0.2. The basic example.** It will be useful throughout the paper to keep the following perspective in mind. Suppose  $X$  classifies some kind of object in the sense that maps  $S \rightarrow X$  form the space of such objects over  $S$ . Then the loop space  $\mathcal{L}X$  classifies families of such objects over the circle  $S^1$ . (It is worth emphasizing that a family of objects over a topological space is by necessity locally constant.) Likewise, the formal loop space  $\widehat{\mathcal{L}}X$  classifies families over  $S^1$  that are infinitesimally close to a trivial family, and the unipotent loop space  $\mathcal{L}^u X$  classifies algebraic one-parameter families.

We illustrate the above notions with the basic motivating example  $X = BG = pt/G$ , the classifying stack of a reductive group  $G$ , so  $X$  classifies principal  $G$ -bundles. We have the elementary identifications of loop spaces

$$\mathcal{L}(BG) \simeq G/G \quad \mathcal{L}^u(BG) \simeq G^u/G \quad \widehat{\mathcal{L}}(BG) \simeq \widehat{G}/G$$

where all of the quotients are with respect to conjugation,  $G^u$  is the formal neighborhood of the unipotent elements of  $G$ , and  $\widehat{G}$  is the formal group of  $G$ . In other words,  $\mathcal{L}BG$  classifies  $G$ -local systems on  $S^1$ . If we trivialize such a local system at a point, its monodromy gives an element of  $G$ . Forgetting the trivialization passes to the adjoint quotient  $G/G$ . Likewise,  $\widehat{\mathcal{L}}BG$  classifies local systems whose monodromy is in the formal neighborhood of the identity, and  $\mathcal{L}^u BG$  classifies local systems with unipotent monodromy.

Recall the characteristic polynomial map

$$\chi : \mathcal{L}(BG) \simeq G/G \longrightarrow H//W$$

where  $H//W = \text{Spec } \mathcal{O}(H)^W$  denotes the affine quotient. By Chevalley's Theorem,  $\chi$  induces an equivalence on global functions

$$\chi^* : \mathcal{O}(H)^W \xrightarrow{\sim} \mathcal{O}(G/G)$$

and thus  $H//W$  is the affinization of  $\mathcal{L}(BG) \simeq G/G$ . From the point of view of local systems, the characteristic polynomial map simply remembers the eigenvalues of the monodromy.

Observe that the identification  $\mathcal{L}(BG) \simeq G/G$  restricts to an identification

$$\mathcal{L}^u(BG) \simeq \chi^{-1}(\widehat{H}//W)$$

where  $\widehat{H}$  denotes the formal group of  $H$ . More generally, fix a semisimple  $\alpha \in G$  with class  $[\alpha] \in H//W$ , and let  $G(\alpha) \subset G$  be the centralizer of  $\alpha$ . Then we have an identification

$$\mathcal{L}^u(BG(\alpha)) \simeq \chi^{-1}(\widehat{H}_{W \cdot \alpha} // W)$$

where  $\widehat{H}_{W \cdot \alpha}$  denotes the formal neighborhood of the Weyl orbit  $W \cdot \alpha \subset H$ .

*Remark 2.2* (Jordan decomposition). We can interpret the above identifications in terms of *Jordan decomposition* of elements of  $G$ . Namely, separating out the semisimple part of elements breaks up the loop space  $\mathcal{L}BG$  into unipotent loop spaces.

One can imitate the Jordan decomposition for the loop space  $\mathcal{L}X$  of a general geometric stack. We will perform this decomposition explicitly for examples associated with flag varieties in Sections 3 and 4. We merely note here that the underlying underived stack of  $\mathcal{L}X$ , the inertia stack  $IX$ , is an affine group scheme over  $X$ , and so we may speak of the semisimple part of any loop  $\gamma \in \mathcal{L}X$  as a point of inertia  $\gamma_{ss} \in IX$ . One may then combine this with the notion of unipotent loop introduced above and extend the Jordan decomposition of inertia to a decomposition of the loop space in terms of unipotent loop spaces of a collection of associated stacks. Combined with the relation of sheaves on unipotent loops to  $\mathcal{D}$ -modules recalled below, this provides an approach to understanding sheaves on the entire loop space.

**2.0.3. Rotating loops.** The circle  $S^1$ , and hence the loop space  $\mathcal{L}X$ , carries a natural rotational  $S^1$ -action. The affinization  $\mathrm{Aff}(S^1) \simeq B\mathbb{G}_a$  of the circle, and hence the unipotent loop space  $\mathcal{L}^u X$ , carries a natural translational  $B\mathbb{G}_a$ -action so that the inclusion  $\mathcal{L}^u X \rightarrow \mathcal{L}X$  is equivariant for the natural homomorphism  $S^1 \rightarrow \mathrm{Aff}(S^1) \simeq B\mathbb{G}_a$ . In addition,  $\mathcal{L}^u X$  carries a compatible action of the multiplicative group  $\mathbb{G}_m$  induced by its action on  $\mathbb{G}_a$ . This  $\mathbb{G}_m$ -action is an expression of the formality of  $\mathcal{O}(S^1) = C^*(S^1, k)$ .

It is shown in [BN2] that the translational  $B\mathbb{G}_a$ -action on  $\mathcal{L}^u X$  is completely determined by the data of a degree one vector field with square zero. Restricted to formal loops, this vector field is identified with the de Rham differential. Let  $\mathbb{T}_X[-1] = \mathrm{Spec}_{\mathcal{O}_X} \Omega_X^\bullet$  denote the odd tangent bundle of  $X$ , and  $\widehat{\mathbb{T}}_X[-1]$  its formal completion along the zero section.

**Theorem 2.3** ([BN2]). *Let  $X$  be a geometric stack. There is a canonical identification*

$$\widehat{\mathcal{L}}X \simeq \widehat{\mathbb{T}}_X[-1]$$

*such that the odd vector field of the  $S^1$ -action of loop rotation corresponds to that given by the de Rham differential.*

*Remark 2.4.* The theorem is a generalization to arbitrary geometric stacks of the exponential map identifying the adjoint quotients of the completed Lie algebra  $\widehat{\mathfrak{g}}/G$  and formal group  $\widehat{G}/G$ .

The above theorem provides a relation between equivariant sheaves on the loop space of a smooth stack and sheaves with flat connection on the stack itself.

Consider the equivariant cohomology ring

$$\mathcal{O}(BS^1) \simeq H^*(BS^1, k) \simeq k[u], \text{ with } u \text{ of degree } 2.$$

The affinization morphism  $S^1 \rightarrow B\mathbb{G}_a$  induces an identification  $\mathcal{O}(BS^1) \simeq \mathcal{O}(B\mathbb{G}_a)$ , and hence there is a natural  $\mathbb{G}_m$ -action on the cohomology such that  $u$  has weight 1.

Define the graded Rees algebra  $\mathcal{R}_X$  to be the sheaf of  $\mathbf{Z}$ -graded  $k$ -algebras

$$\mathcal{R}_X = \bigoplus_{i \geq 0} u^i \mathcal{D}_X^{\leq i} \subset \mathcal{D}_X \otimes_k k[u]$$

where  $\mathcal{D}_X^{\leq i} \subset \mathcal{D}_X$  denotes differential operators of order at most  $i$ . Note that  $\mathcal{R}_X$  is concentrated in positive cohomological degree twice the weight degree.

Let  $\mathcal{R}_X\text{-cmod}_{\mathbf{Z}}$  denote the stable  $\infty$ -category of graded quasicoherent sheaves on  $X$  equipped with the compatible structure of graded complete  $\mathcal{R}_X$ -module.

**Theorem 2.5** ([BN2]). *For  $X$  a smooth geometric stack, there is a canonical equivalence of stable  $\infty$ -categories*

$$\mathrm{QCoh}(\widehat{\mathcal{L}}X)^{\mathbb{S}} \simeq \mathcal{R}_X\text{-cmod}_{\mathbf{Z}}.$$

*Remark 2.6.* The classical Rees algebra  $\mathcal{R}_X^{cl}$  is positively graded by weight but lives in cohomological degree zero. As explained in [BN2], there is a canonical shear equivalence

$$\mathcal{R}_X^{cl}\text{-cmod}_{\mathbf{Z}} \simeq \mathcal{R}_X\text{-cmod}_{\mathbf{Z}}.$$

To recover  $\mathcal{D}_X$ -modules from  $\mathcal{R}_X$ -modules, we must localize. On the one hand, any category of  $S^1$ -equivariant (or  $B\mathbb{G}_a$ -equivariant) sheaves such as  $\mathrm{QCoh}(\widehat{\mathcal{L}}X)^{\mathbb{S}}$  is linear over the equivariant cohomology ring  $\mathcal{O}(BS^1) \simeq \mathcal{O}(B\mathbb{G}_a) \simeq k[u]$ . On the other hand, by construction, the algebra  $k[u]$  maps to the graded Rees algebra  $\mathcal{R}_X$  as a central subalgebra.

When we invert  $u$  and localize the categories, we will denote the result by the subscript *loc*. So for a compatibly graded  $k[u]$ -linear stable  $\infty$ -category  $\mathcal{C}$ , we have the new stable  $\infty$ -category

$$\mathcal{C}_{loc} = \mathcal{C} \otimes_{k[u]} k[u, u^{-1}].$$

If  $\mathcal{C}$  were not graded,  $\mathcal{C}_{loc}$  would be only  $\mathbf{Z}/2\mathbf{Z}$ -graded since  $u$  is of the cohomological degree 2. But when we localize a compatibly graded category, the localized category maintains a usual cohomological degree. The natural projection  $p : \mathcal{C} \rightarrow \mathcal{C}_{loc}$  satisfies

$$p(c[-2]\langle -1 \rangle) = p(c)$$

where  $[-2]$  denotes the cohomological shift by two, and  $\langle -1 \rangle$  denotes the weight shift by one.

Now to describe the localization of the equivalence of the previous theorem, we introduce the following further definitions.

For a graded  $\infty$ -category, we can consider the full  $\infty$ -subcategory of objects with weights bounded from above. In particular, in the case of  $\mathbb{S}$ -invariants in an  $\infty$ -category  $\mathcal{C}$  with  $\mathbb{S}$ -action, we write  $\mathcal{C}^{\mathbb{S}-}$  to denote the  $\mathbb{S}$ -invariants with bounded above weight. As explained in [BN2], such a weight bound allows one to work with unipotent loops rather than formal loops thanks to the canonical equivalence

$$\mathrm{QCoh}(\mathcal{L}^u X)^{\mathbb{S}-} \simeq \mathrm{QCoh}(\widehat{\mathcal{L}} X)^{\mathbb{S}-}.$$

Recall that by a  $\mathcal{D}$ -module on a smooth stack  $X$  we will always mean a *reasonable*  $\mathcal{D}$ -module, i.e. one that admits a reasonable filtration, and so arises in the localization of the category of complete graded Rees modules. Reasonable  $\mathcal{D}$ -modules admit the following alternative description. Given an object  $M$  of the dg derived category  $\mathcal{D}_X\text{-mod}$  of all  $\mathcal{D}$ -modules on  $X$ , consider its de Rham cohomology  $r(M) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, M)$ . Let  $\mathcal{K}er(r)$  denote the full dg subcategory of  $\mathcal{D}_X\text{-mod}$  given by the kernel of  $r$ . We can define the dg derived category  $\mathcal{D}(X)$  of reasonable  $\mathcal{D}$ -modules to be the full dg subcategory of  $\mathcal{D}_X\text{-mod}$  given by the left orthogonal  ${}^\perp \mathcal{K}er(r)$ . One can show  ${}^\perp \mathcal{K}er(r)$  is equivalent to the dg quotient  $\mathcal{D}_X\text{-mod} / \mathcal{K}er(r)$ .

Note that all coherent  $\mathcal{D}$ -modules are reasonable. It is simple to describe the unreasonable  $\mathcal{D}$ -modules given by  $\mathcal{K}er(r)$ . Let  $\mathcal{E}_X$  denote the algebra obtained by microlocalizing  $\mathcal{D}_X$  away from the zero section. (In a local coordinate chart,  $\mathcal{D}_X$  is the Weyl algebra  $k\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n} \rangle$ , and  $\mathcal{E}_X$  is the localized Weyl algebra  $k\langle x_1, \dots, x_n, \partial_{x_1}, \partial_{x_1}^{-1}, \dots, \partial_{x_n}, \partial_{x_n}^{-1} \rangle$ .) There is a natural pushforward functor  $\mathcal{E}_X\text{-mod} \rightarrow \mathcal{D}_X\text{-mod}$ , and  $\mathcal{K}er(r)$  is precisely its image.

In the rest of the paper, the only result of [BN2] we will cite is the following.

**Corollary 2.7** ([BN2]). *There is a canonical equivalence of stable  $\infty$ -categories*

$$\mathrm{QCoh}(\mathcal{L}^u X)_{loc}^{\mathbb{S}-} \simeq \mathcal{D}(X).$$

### 3. STEINBERG VARIETIES AS LOOP SPACES

In this section, we explain how one can view equivariant Steinberg varieties as loop spaces. The entire equivariant Grothendieck-Steinberg variety is a loop space, and its fixed-monodromy subspaces are unipotent loop spaces.

**3.1. The Grothendieck-Springer resolution via loops.** We begin with a warm-up, identifying the Grothendieck-Springer resolution in terms of loop spaces. We follow the blueprint of our basic example  $BG$ , discussed in Section 2.0.2 above.

Let  $BB = pt/B$  denote the classifying stack of a Borel subgroup  $B \subset G$ . We have the elementary identifications of loop spaces

$$\mathcal{L}(BB) \simeq B/B \quad \mathcal{L}^u(BB) \simeq B^u/B \quad \widehat{\mathcal{L}}(BB) \simeq \widehat{B}/B$$

Recall the Grothendieck-Springer simultaneous resolution

$$\mathcal{B} \xleftarrow{\pi} \widetilde{G} = \{(g, B) \in G \times \mathcal{B} \mid \mathrm{Ad}_g(B) = B\} \xrightarrow{\mu} G.$$

We have the ordered eigenvalue map

$$\tilde{\chi} : \tilde{G} \longrightarrow H \quad \tilde{\chi}(g, B) = [g] \in B/U \simeq H.$$

The fiber  $\tilde{\chi}^{-1}(e) \subset \tilde{G}$  over the identity  $e \in H$  is the traditional Springer resolution where the group element  $g$  is required to be unipotent.

Observe that there is a natural identification

$$\mathcal{L}(BB) \simeq B/B \simeq \tilde{G}/G$$

where the quotients are with respect to conjugation. Since the ordered eigenvalue map  $\tilde{\chi}$  is invariant under conjugation, it descends to the adjoint quotient

$$\tilde{\chi} : \tilde{G}/G \longrightarrow H$$

This map in turn induces an equivalence on global functions

$$\tilde{\chi}^* : \mathcal{O}(H) \xrightarrow{\sim} \mathcal{O}(\tilde{G}/G)$$

and thus  $H$  is the affinization of  $\mathcal{L}(BB) \simeq \tilde{G}/G$ .

*Remark 3.1.* We can view  $\mathcal{L}(BB) \simeq B/B$  as  $B$ -local systems on the circle  $S^1$ . Of course, it is obviously equivalent to  $G$ -local systems with a  $B$ -reduction. Alternatively, it is also equivalent to  $G$ -local systems on  $S^1$  with a monodromy-invariant  $B$ -reduction at a point of  $S^1$ . If we trivialize the  $G$ -local system at the point, we obtain an element of  $\tilde{G}$ .

**3.2. Loops in flag varieties.** We will use the name Grothendieck-Steinberg variety  $St$  for the fiber product

$$St = \tilde{G} \times_G \tilde{G} = \{(g, B_1, B_2) \in G \times \mathcal{B} \times \mathcal{B} \mid g \in B_1 \cap B_2\}.$$

Note that because the projection  $\tilde{G} \rightarrow G$  is a semi-small resolution, the derived fiber product coincides with the naive fiber product. In general,  $St$  is connected, but has irreducible components labeled by the Weyl group  $W$ . Hence if  $G$  is noncommutative,  $St$  is singular.

The following is the fundamental observation that motivated this paper.

**Theorem 3.2.** *We have a canonical identification of the loop space*

$$\mathcal{L}(B \backslash G/B) \simeq St/G.$$

*Proof.* Recall that by definition, the inertia stack  $I((\mathcal{B} \times \mathcal{B})/G)$  is the underived mapping stack  $\text{Map}_{St}(S^1, (\mathcal{B} \times \mathcal{B})/G)$ . It is immediate from the definitions that  $I((\mathcal{B} \times \mathcal{B})/G)$  is precisely the global Grothendieck-Steinberg space  $St/G$ . Thus to establish the theorem, we must see that the loop space  $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G)$  coincides with  $I((\mathcal{B} \times \mathcal{B})/G)$ . In other words, we must see that the derived structure of  $\mathcal{L}((\mathcal{B} \times \mathcal{B})/G)$  is trivial. This is implied by Proposition 3.3 immediately following.  $\square$

**Proposition 3.3.** *Suppose  $X = Y/G$ , where  $Y$  is a quasi-projective variety and  $G$  is affine. Then the loop space  $\mathcal{L}X$  has trivial derived structure if and only if there are finitely many  $G$ -orbits in  $Y$ .*

*Proof.* We represent  $X$  as a groupoid scheme with  $X_0 = Y$  as scheme of objects (0-simplices) and  $X_1 = G \times Y$  as scheme of morphisms (1-simplices). It is convenient to rewrite the pushforward to  $X$  of the structure sheaf  $\mathcal{O}_{\mathcal{L}X}$  as the descent of the derived tensor product

$$\mathcal{O}_{X_1} \otimes_{X_0 \times X_1}^{\mathbf{L}} \mathcal{O}_{X_1}$$

where  $X_1$  maps to  $X_0 \times X_1$  by the product maps  $\ell \times \text{id}_{X_1}$  and  $r \times \text{id}_{X_1}$ . This can be viewed as the structure sheaf of the derived intersection of the subschemes  $\Gamma_\ell, \Gamma_r \subset X_0 \times X_1$  given by

the graphs of  $\ell, r$  respectively. Here loops are thought of as pairs of 1-simplices that are equal and such that the left end of the first is glued to the right end of the second.

Now our assertion will follow from a simple dimension count. Let  $n_0$  and  $n_1$  be the dimensions of  $X_0$  and  $X_1$  respectively. On the one hand, the expected dimension of the intersection  $\Gamma_\ell \cap \Gamma_r$  inside of  $X_0 \times X_1$  is given by  $n_1 + n_1 - (n_0 + n_1) = n_1 - n_0$ . On the other hand, each isomorphism class of objects of  $X$  contributes a subscheme of precisely dimension  $n_1 - n_0$  to the intersection. Thus the intersection has the expected dimension if and only if there is no nontrivial moduli of isomorphism classes of objects.  $\square$

*Remark 3.4.* We can view  $B \backslash G/B \simeq BB \times_{BG} BB$  as the moduli of  $G$ -local systems on the interval  $[0, 1]$  with  $B$ -reductions at the end points  $\{0, 1\}$ . Similarly, we can view the equivariant Grothendieck-Steinberg variety  $\mathcal{L}(B \backslash G/B) \simeq St/G$  as the moduli of  $G$ -local systems on the cylinder  $Cyl = [0, 1] \times S^1$  with  $B$ -reductions at the boundary circles  $\{0, 1\} \times S^1$ . The fact that taking loops commutes with forming fiber products (both are limits), implies the identification  $St_G/G \simeq \tilde{G}/G \times_{G/G} \tilde{G}/G$ .

**3.3. Fixed monodromicity.** We have two copies of the ordered eigenvalue map

$$\tilde{\chi} : St \longrightarrow H \times H \quad \tilde{\chi}(g, B_1, B_2) = ([g]_1, [g]_2) \in B_1/U_1 \times B_2/U_2 \simeq H \times H.$$

The fiber  $\tilde{\chi}^{-1}(e, e) \subset St$  over the identity  $(e, e) \in H \times H$  is the usual Steinberg variety, where the group element  $g$  is required to be unipotent.

The image of  $\tilde{\chi}$  consists of pairs of elements  $(\alpha, \beta) \in H \times H$  related by the Weyl group action:  $\beta = w \cdot \alpha$ , for some  $w \in W$ . In other words, it consists of the union

$$\mathcal{H} = \coprod_{w \in W} \{(h, w \cdot h) | h \in H\} \subset H \times H$$

of the graphs of the automorphisms of  $H$  given by Weyl group elements.

Since the ordered eigenvalue map  $\tilde{\chi}$  is invariant under conjugation, it descends to the adjoint quotient

$$\tilde{\chi} : St/G \longrightarrow \mathcal{H}.$$

This map in turn realizes  $\mathcal{H}$  as the affinization of  $St/G$ .

Fix an element  $(\alpha, w \cdot \alpha) \in \mathcal{H}$ , and let  $\hat{\mathcal{H}}_{\alpha, w \cdot \alpha}$  be its formal neighborhood. We will use the name monodromic Steinberg variety for the inverse image

$$St_{\alpha, w \cdot \alpha} = \tilde{\chi}^{-1}(\hat{\mathcal{H}}_{\alpha, w \cdot \alpha})$$

Let  $\mathcal{O}_\alpha \subset G$  denote the semisimple conjugacy class corresponding to  $\alpha$ . Fix once and for all an element  $\tilde{\alpha} \in \mathcal{O}_\alpha$ , and let  $G(\alpha) \subset G$  denote its centralizer. In general,  $G(\alpha)$  is reductive of the same rank as  $G$ , and often turns out to be a Levi subgroup.

We will affix the symbol  $(\alpha)$  to our usual notation when referring to objects associated to  $G(\alpha)$ . So for example, we write  $B(\alpha) \subset G(\alpha)$  for a Borel subgroup.

The following key variation on Theorem 3.2 further justifies the importance of loop spaces to Langlands parameters for representations.

**Theorem 3.5.** *For any  $\alpha \in H$ , and  $w \in W$ , we have a canonical identification of the unipotent loop space and monodromic Steinberg variety*

$$\mathcal{L}^u(B(\alpha) \backslash G(\alpha)/B(\alpha)) \simeq St_{\alpha, w \cdot \alpha}/G$$

*Proof.* First, note that multiplication by the central element  $\tilde{\alpha} \in G(\alpha)$  provides an equivalence

$$\mathcal{L}^u(B(\alpha) \backslash G(\alpha)/B(\alpha)) \simeq St(\alpha)_{e, e} \xrightarrow{\sim} St(\alpha)_{\alpha, \alpha}$$

where  $e \in H$  is the identity.

Now for each  $w \in W$ , one can check that there is a map

$$\mathcal{St}(\alpha)_{\alpha, \alpha} \longrightarrow \mathcal{St}_{\alpha, w\alpha} \quad (g, B(\alpha)_1, B(\alpha)_2) \longmapsto (g, B_1, B_2)$$

uniquely characterized by the properties:

$$\begin{aligned} B_1 \cap G(\alpha) &= B(\alpha)_1 & B_2 \cap G(\alpha) &= B(\alpha)_2 \\ [g]_1 &\in \widehat{H}_\alpha & [g]_2 &\in \widehat{H}_{w\alpha} \end{aligned}$$

where  $[g]_1, [g]_2$  denote the classes of  $g$  in  $B_1/U_1, B_2/U_2$  respectively.

Passing to the respective quotients gives the sought-after isomorphism.  $\square$

An interesting aspect of the theorem is the general “discontinuity” of the objects appearing on the left hand side. From a geometric perspective, the quotients  $B(\alpha) \backslash G(\alpha) / B(\alpha)$  do not form a nice family as we vary the parameter  $\alpha$ . But the theorem says that the loop spaces of these quotients do fit into the nice family  $\tilde{\chi} : \mathcal{St}/G \rightarrow \mathcal{H} \subset H \times H$ . Here we should emphasize that we are thinking about  $\mathcal{St}/G$  as a loop space, rather than along the more traditional lines identifying the usual Steinberg variety (i.e.,  $\tilde{\chi}^{-1}(e, e)$ ) with a union of conormals (or equivariantly, as the cotangent bundle to  $B \backslash G/B$ ). In the discussion to follow, we describe similar results for geometric parameter spaces for real reductive groups. In that context, it is only the loop spaces that fit together into a nice family, not the cotangent bundles.

**3.4. Application to Hecke categories.** Let us focus on the most interesting case of trivial monodromicity when  $\alpha$  is the identity  $e \in H$ . By Theorem 3.5, we have a canonical identification

$$\mathcal{L}^u(B \backslash G/B) \simeq \mathcal{St}_{e,e}/G$$

where the Steinberg variety  $\mathcal{St}^u = \mathcal{St}_{e,e}$  is nothing more than the fiber product

$$\mathcal{St}^u = \tilde{G}^u \times_{G^u} \tilde{G}^u = \{(g, B_1, B_2) \in G^u \times \mathcal{B} \times \mathcal{B} \mid g \in B_1 \cap B_2\},$$

where  $G^u \subset G$  is the formal neighborhood of the unipotent elements, and  $\tilde{G}^u$  is its Springer resolution

$$\tilde{G}^u = \{(g, B) \in G^u \times \mathcal{B} \mid g \in B\}.$$

Now our results from [BN2], as recalled in Section 2, immediately provide a canonical equivalence of  $\infty$ -categories

$$\mathrm{QCoh}(\mathcal{St}^u/G)_{loc}^{\mathbb{S}-} \simeq \mathcal{D}(B \backslash G/B).$$

Observe that on the one hand, the underlying dg category  $\mathrm{QCoh}(\mathcal{St}^u/G)$  is the affine Hecke category  $\mathcal{H}_G^{aff}$ . On the other hand, the dg category  $\mathcal{D}(B \backslash G/B)$  is the finite Hecke category  $\mathcal{H}_G$ .

Thus we arrive at the following fundamental relationships.

**Corollary 3.6.** *The localized  $\mathbb{S}$ -invariants with bounded above weight in the affine Hecke category  $\mathcal{H}_G^{aff}$  are canonically equivalent to the finite Hecke category  $\mathcal{H}_G$ .*

For general monodromicity  $\alpha \in H$ , again by Theorem 3.5, we have a canonical identification

$$\mathcal{L}^u(B(\alpha) \backslash G(\alpha) / B(\alpha)) \simeq \mathcal{St}_{\alpha, w\alpha}/G$$

Thus we can transport the canonical  $\mathbb{S}$ -action on the unipotent loop space of the left hand side to the monodromic Steinberg variety of the right hand side.

**Corollary 3.7.** *For general  $\alpha \in H$ , we have a canonical equivalence*

$$\mathrm{QCoh}(\mathcal{St}_{\alpha, w\alpha}/G)_{loc}^{\mathbb{S}-} \simeq \mathcal{D}(B(\alpha) \backslash G(\alpha) / B(\alpha)).$$

*Remark 3.8* (Comparison of circle actions). By Theorem 3.2, we have a canonical identification

$$\mathcal{L}(B \backslash G / B) \simeq St / G,$$

and it is evident that the rotational  $S^1$ -action on the loop space of the left hand side restricts to an  $S^1$ -action on the substack

$$St_{\alpha, w\alpha} / G \subset St / G.$$

On the other hand, in Corollary 3.7, we used an  $\mathbb{S}$ -action, and in particular an  $S^1$ -action, on  $St_{\alpha, w\alpha} / G$  coming from its identification as a unipotent loop space

$$St_{\alpha, w\alpha} / G \simeq \mathcal{L}^u(B(\alpha) \backslash G(\alpha) / B(\alpha)).$$

To distinguish the two actions, we will write  $S^1_\alpha$  for the former circle, and  $S^1$  for the latter.

It is natural to try to relate these two circle actions and the corresponding categories of equivariant sheaves. We restrict ourselves to an informal discussion of the following assertion: while the two  $S^1$ -actions differ, the corresponding localized (2-periodic) categories of  $S^1$ -equivariant coherent sheaves are equivalent:

$$\mathrm{Coh}(St_{\alpha, w\alpha} / G)^{S^1_\alpha}_{loc} \simeq \mathrm{Coh}(St_{\alpha, w\alpha} / G)^{S^1}_{loc}.$$

To justify this claim, in parallel to Corollary 3.7, we will relate each category to suitable  $\mathcal{D}$ -modules on  $B(\alpha) \backslash G(\alpha) / B(\alpha)$ , with localized (2-periodic) coefficients.

To begin, let us comment on our reason for considering coherent sheaves as opposed to all quasicoherent sheaves. The action of  $\mathbb{G}_m$ , and hence  $\mathbb{S}$ , on the unipotent loop space  $St^u / G$  does not extend to the full loop space  $St / G$ , or in particular the piece  $St_{\alpha, w\alpha} / G$ . But to reach  $\mathcal{D}$ -modules, we need that the restriction of localized equivariant sheaves along the inclusion

$$\widehat{\mathcal{L}}(B(\alpha) \backslash G(\alpha) / B(\alpha)) \hookrightarrow \mathcal{L}^u(B(\alpha) \backslash G(\alpha) / B(\alpha)) \simeq St_{\alpha, w\alpha} / G$$

is an equivalence. For this, we need some alternative to our usual  $\mathbb{G}_m$ -weight bounds. But the maximal torus in the symmetry group  $G(\alpha)$  sufficiently rescales elements so that the restriction of coherent sheaves is an equivalence. In the context of the above claim, the upshot is that we can identify  $\mathrm{Coh}(St_{\alpha, w\alpha} / G)^{S^1_\alpha}_{loc}$  with a suitable category of  $\mathcal{D}$ -modules on  $B(\alpha) \backslash G(\alpha) / B(\alpha)$ , with localized (2-periodic) coefficients.

Now to justify the above claim, note that an action of  $S^1 \simeq B\mathbb{Z}$  on an  $\infty$ -category induces an automorphism of the identity functor. We will refer to it as the universal monodromy (and ignore the higher coherences required to fully specify an  $S^1$ -action). Let us denote by  $m_\alpha$  the universal monodromy of the  $S^1_\alpha$ -action on  $\mathrm{Coh}(St_{\alpha, w\alpha} / G)$ , and by  $m_e$  the universal monodromy of the  $S^1$ -action. Recall that a key step in the proof of Theorem 3.5 was the equivalence

$$St(\alpha)_{e, e} / G(\alpha) \xrightarrow{\sim} St(\alpha)_{\alpha, \alpha} / G(\alpha)$$

given by multiplication by the central element  $\alpha \in G(\alpha)$ . It follows that we have an identity

$$m_\alpha = m_e \circ \varphi_\alpha$$

where  $\varphi_\alpha$  is the automorphism of the identity functor of  $\mathrm{Coh}(St_{\alpha, w\alpha} / G)$  given by the central element  $\alpha \in G(\alpha)$ .

The fact that  $\alpha$  is central implies that  $\varphi_\alpha$  commutes with the universal monodromy  $m_e$ . In particular,  $\varphi_\alpha$  passes to the  $S^1$ -equivariant category  $\mathrm{Coh}(St_{\alpha, w\alpha} / G)^{S^1}$ , and further to its (2-periodic) localization  $\mathrm{Coh}(St_{\alpha, w\alpha} / G)^{S^1}_{loc}$ . But after localization, the action of  $\varphi_\alpha$  is trivial (in fact, can be unambiguously trivialized). To see this, recall that we have identified  $\mathrm{Coh}(St_{\alpha, w\alpha} / G)^{S^1}_{loc}$  with suitable  $\mathcal{D}$ -modules on  $B(\alpha) \backslash G(\alpha) / B(\alpha)$ , with localized (2-periodic) coefficients, and the connected components of automorphism groups act trivially on  $\mathcal{D}$ -modules on a stack. Furthermore, a fixed choice of logarithm  $\lambda = \log(\alpha)$  determines a path  $\exp(t\lambda)$

from the identity  $e \in G(\alpha)$  to the central element  $\alpha$ . Hence this path provides a canonical trivialization of the automorphism  $\varphi_\alpha$ . Thus we conclude that  $\mathrm{Coh}(\mathcal{S}t_{\alpha, w\alpha}/G)_{loc}^{S^1}$  is equivalent to the localized equivariant category  $\mathrm{Coh}(\mathcal{S}t_{\alpha, w\alpha}/G)_{loc}^{S^1_\alpha}$  for the alternative circle action.

#### 4. LANGLANDS PARAMETERS AS LOOP SPACES

Motivated by an ongoing project to better understand representations of real groups, we were led to a Galois-twisted version of the relationship between loop spaces and Steinberg varieties developed in the previous section.

**4.1. Involutions of flag varieties.** Let  $\Gamma$  be a finite group.

A  $\Gamma$ -action on a derived stack  $Y$  is by definition a derived stack  $\mathcal{Y} \rightarrow B\Gamma$  equipped with an identification  $\mathcal{Y} \times_{B\Gamma} E\Gamma \simeq Y$ . We will usually write  $Y/\Gamma$  in place of  $\mathcal{Y}$ . The  $\Gamma$ -invariants (or  $\Gamma$ -fixed points) of  $Y$  are by definition the derived mapping stack of sections

$$Y^\Gamma = \mathrm{Map}_{B\Gamma}(B\Gamma, Y/\Gamma).$$

(Note that the above discussion applies to  $\Gamma$  an arbitrary group derived stack.)

We will encounter the situation of two different  $\Gamma$ -actions on single  $Y$ . To avoid confusion, we will consider two copies  $\Gamma$  and  $\Gamma'$  of the group indexed by the two actions, and write  $Y^\Gamma$  and  $Y^{\Gamma'}$  to denote the invariants of the respective actions.

Let us specialize to  $\Gamma$  the cyclic group  $\mathbf{Z}/2\mathbf{Z}$ .

Consider the  $\Gamma$ -action on  $G$  and in turn  $BG$  induced by an involution  $\eta$  of  $G$ . Note that the classifying space construction and group invariants do not necessarily commute (the former is a colimit and the latter is a limit). There is always a map  $B(G^\Gamma) \rightarrow (BG)^\Gamma$ , but in general  $(BG)^\Gamma$  is a far richer object.

**Theorem 4.1.** *There is a finite set  $I$  of involutions of  $G$  containing  $\eta$  and a natural identification*

$$(BG)^\Gamma \simeq \coprod_{\iota \in I} BK_\iota$$

where  $K_\iota \subset G$  denotes the fixed-point subgroup of the involution  $\iota$ .

*Proof.* Let  $G \rtimes \Gamma$  be the semidirect product with respect to the given action. Unwinding the definitions gives that  $(BG)^\Gamma$  is equivalent to homomorphisms  $\Gamma \rightarrow G \rtimes \Gamma$ , up to conjugation, such that the composition  $\Gamma \rightarrow G \rtimes \Gamma \rightarrow \Gamma$  is the identity. Thus we have a natural identification

$$(BG)^\Gamma \simeq \{g \in G \mid g\eta(g) = 1\}/G$$

We take  $I$  to be the set of components of this stack – the set of conjugacy classes of solutions to the above equation – and pick one  $\iota$  from each class.  $\square$

*Remark 4.2.* We can view the quotient  $\mathcal{G}_{B\Gamma} = G/\Gamma$  as a group scheme over  $B\Gamma$  such that its base change along  $E\Gamma \rightarrow B\Gamma$  is identified with  $G$ . From this perspective, the invariants  $(BG)^\Gamma$  form the moduli of principal  $\mathcal{G}_{B\Gamma}$ -bundles over  $B\Gamma$ . Forgetting the invariant structure on such a bundle corresponds to base change along  $E\Gamma \rightarrow B\Gamma$ .

Now consider the  $\Gamma$ -action on the fiber product

$$B \backslash G/B \simeq BB \times_{BG} BB \simeq G \backslash (G/B \times G/B)$$

induced by the involution  $\eta$  and the exchange of the two factors. As an immediate consequence of Theorem 4.1, we can calculate the  $\Gamma$ -fixed points in  $B \backslash G/B$  in terms of quotients of  $G/B$  by symmetric subgroups.



**Corollary 4.3.** *There is a natural identification*

$$(B \backslash G/B)^\Gamma \simeq \coprod_{\iota \in I} K_\iota \backslash G/B.$$

*Remark 4.4.* Recall that we can view  $B \backslash G/B \simeq BB \times_{BG} BB$  as the moduli of  $G$ -local systems on the interval  $[0, 1]$  with  $B$ -reductions at the end points  $\{0, 1\}$ . From this perspective, flipping  $[0, 1]$  around the midpoint  $1/2 \in [0, 1]$  induces the involution that exchanges the two factors of the fiber product.

Furthermore, the involution  $\eta$  descends the group  $G$  to a group scheme  $\mathcal{G}_{[0, 1/2]} = (G \times [0, 1])/\Gamma$  over the quotient  $[0, 1/2] = [0, 1]/\Gamma$ . Finally, the invariants  $(B \backslash G/B)^\Gamma$  form the moduli of  $\mathcal{G}_{[0, 1/2]}$ -local systems on  $[0, 1/2]$  with  $B$ -reduction at the boundary point  $\{0\}$ .

As our notation suggests, we often regard  $[0, 1/2]$  as the interval  $[0, 1/2]$  with the end point  $\{1/2\}$  equipped with the symmetries  $\Gamma$ . This makes apparent the identification of Corollary 4.3 via the intermediate realization  $(B \backslash G/B)^\Gamma \simeq (BG)^\Gamma \times_{BG} BB$ .

**4.2. Involutions of Steinberg varieties.** We continue to let  $\Gamma$  denote the cyclic group  $\mathbf{Z}/2\mathbf{Z}$ .

**Definition 4.5.** The *Langlands parameter variety*  $\mathcal{L}$  is defined to be

$$\mathcal{L} = \{(g, B) \in G \times \mathcal{B} \mid g\eta(g) \in B\}.$$

Consider the involution of the Grothendieck-Steinberg variety

$$St/G \simeq \mathcal{L}(BB) \times_{\mathcal{L}(BG)} \mathcal{L}(BB)$$

induced by the involution  $\eta$  of  $BG$ , the exchange of the two factors of  $\mathcal{L}(BB)$ , and the antipodal map on  $S^1$ . To reflect the fact that we are twisting by the antipodal map on  $S^1$ , we will write  $\Gamma_{tw}$  for the resulting  $\mathbf{Z}/2\mathbf{Z}$ -action on  $St$ .

**Theorem 4.6.** *There is a natural identification*

$$\mathcal{L}/G \simeq (St/G)^{\Gamma_{tw}} \simeq (\mathcal{L}(B \backslash G/B))^{\Gamma_{tw}}.$$

*Proof.* This is a straightforward descent assertion from  $G$ -local systems on a cylinder with  $B$ -reductions along the two boundary circles to  $\eta$ -twisted  $G$ -local systems on the Möbius strip with  $B$ -reduction along the single boundary circle.  $\square$

*Remark 4.7.* Observe that  $\mathcal{L}/G$  is equipped with a canonical  $S^1$ -action such that the identification of the above theorem is  $S^1$ -equivariant.

*Remark 4.8.* Recall that we can view the equivariant Grothendieck-Steinberg variety  $St/G \simeq \mathcal{L}(B \backslash G/B)$  as the moduli of  $G$ -local systems on the cylinder  $Cyl = [0, 1] \times S^1$  with  $B$ -reductions at the boundary circles  $\{0, 1\} \times S^1$ . From this perspective, the above involution of the cylinder  $Cyl$  has quotient the Möbius strip  $Moeb = ([0, 1] \times S^1)/\Gamma_{tw}$ .

Furthermore, the involution  $\eta$  descends the group  $G$  to a group scheme  $\mathcal{G}_{Moeb} = (G \times Cyl)/\Gamma_{tw}$  over the Möbius strip  $Moeb$ . We conclude that the invariants  $\mathcal{L}/G \simeq (St/G)^{\Gamma_{tw}}$  form the moduli of  $\mathcal{G}_{Moeb}$ -local systems on  $Moeb$  with  $B$ -reduction along the boundary circle  $\{0\} \times S^1$ .

**4.3. Fixed monodromicity.** We have the ordered eigenvalue map

$$\tilde{\chi} : \mathcal{L} \longrightarrow H \quad \tilde{\chi}(g, B) = [g\eta(g)] \in B/U \simeq H.$$

For  $\alpha \in H$ , let  $\hat{H}_\alpha$  denote its formal neighborhood.

**Definition 4.9.** The *monodromic Langlands parameter variety*  $\mathcal{L}_\alpha$  is defined to be

$$\mathcal{L}_\alpha = \tilde{\chi}^{-1}(\hat{H}_\alpha).$$

Note that  $g\eta(g)$  and  $\eta(g\eta(g)) = \eta(g)g$  are conjugate by  $g$ . Thus  $\alpha = [g\eta(g)]$  and  $\eta(\alpha)$  are conjugate by  $w \in W$ .

Consider the  $\Gamma_{tw}$ -action on

$$\mathcal{S}t_{\alpha, \eta(\alpha)}/G \subset \mathcal{S}t/G$$

obtained by restriction.

**Lemma 4.10.** *Fixing monodromicity commutes with taking  $\Gamma_{tw}$ -invariants:*

$$\mathcal{L}_\alpha/G \simeq (\mathcal{S}t_{\alpha, \eta(\alpha)}/G)^{\Gamma_{tw}}.$$

*Proof.* Immediate from definitions.  $\square$

Now we arrive at the main result explaining the relation between Langlands parameters and loop spaces. It is an immediate consequence of our previous results and the following general observation.

Let  $Y$  be a derived stack with  $\Gamma$ -action, and let  $\Gamma$  be the induced action on  $\mathcal{L}Y$ . Let  $\Gamma_{tw}$  be the action on  $\mathcal{L}Y$  obtained by twisting  $\Gamma$  by the antipodal map on  $S^1$ .

**Lemma 4.11.** *There are natural identifications*

$$(\mathcal{L}^u Y)^\Gamma \simeq \mathcal{L}^u(Y^\Gamma) \quad (\mathcal{L}^u Y)^\Gamma \simeq (\mathcal{L}^u Y)^{\Gamma_{tw}}$$

*Proof.* The first assertion is obvious from standard adjunctions.

For the second assertion, by standard adjunctions, we have equivalences

$$(\mathcal{L}Y)^\Gamma \simeq \mathrm{Map}_\Gamma(E\Gamma, \mathcal{L}Y) \simeq \mathrm{Map}_\Gamma(E\Gamma, \mathrm{Map}(S^1, Y)) \simeq \mathrm{Map}_\Gamma(S^1, Y)$$

and similarly for  $\Gamma_{tw}$ .

Substituting  $B\mathbb{G}_a$  for  $S^1$  gives a similar expression for fixed points of unipotent loops. But the antipodal action on  $S^1$  induces the trivial action on  $B\mathbb{G}_a$ . To see this, observe that the square map  $sqr : S^1 \rightarrow S^1$  is the quotient by the antipodal involution, and the induced pullback  $sqr^* : \mathcal{O}(S^1) \rightarrow \mathcal{O}(S^1)$  is the identity on  $\mathcal{O}(S^1) \simeq H^*(S^1, k)$ . Thus the induced map on the affinization  $B\mathbb{G}_a \simeq \mathrm{Aff}(S^1) = \mathrm{Spec} \mathcal{O}(S^1)$  is an equivalence.  $\square$

**Theorem 4.12.** *There is a natural identification*

$$\mathcal{L}_\alpha/G \simeq \mathcal{L}^u((B(\alpha) \backslash G(\alpha)/B(\alpha))^\Gamma)$$

*Proof.* Follows immediately from Theorem 3.5, Lemma 4.10, and Lemma 4.11:

$$\begin{aligned} \mathcal{L}_\alpha/G &\simeq (\mathcal{S}t_{\alpha, \eta(\alpha)}/G)^{\Gamma_{tw}} \\ &\simeq (\mathcal{L}^u(B(\alpha) \backslash G(\alpha)/B(\alpha)))^{\Gamma_{tw}} \\ &\simeq (\mathcal{L}^u(B(\alpha) \backslash G(\alpha)/B(\alpha)))^\Gamma \\ &\simeq \mathcal{L}^u((B(\alpha) \backslash G(\alpha)/B(\alpha))^\Gamma). \end{aligned}$$

$\square$

**Corollary 4.13.** *There is a natural identification*

$$\mathcal{L}_\alpha/G \simeq \coprod_{\iota \in I} \mathcal{L}^u(K(\alpha)_\iota \backslash G(\alpha)/B(\alpha)).$$

*Proof.* Follows immediately from Corollary 4.3 and Theorem 4.12:

$$\mathcal{L}_\alpha/G \simeq \mathcal{L}^u((B(\alpha) \backslash G(\alpha)/B(\alpha))^\Gamma) \simeq \coprod_{\iota \in I} \mathcal{L}^u(K(\alpha)_\iota \backslash G(\alpha)/B(\alpha)).$$

$\square$

*Remark 4.14.* For simplicity, let us focus on the assertion of Theorem 4.12 when  $\alpha$  is the identity  $e \in H$ . Outside of relatively simple deductions, the argument boils down to the identification

$$(\mathcal{L}^u(B \backslash G/B))^\Gamma \simeq (\mathcal{L}^u(B \backslash G/B))^{\Gamma_{tw}}.$$

We can view this as an equivalence between twisted *unipotent* local systems on the product  $[0, 1/2] \times S^1 = ([0, 1]/\Gamma) \times S^1$  and on the Möbius strip  $Moeb = ([0, 1] \times S^1)/\Gamma_{tw}$ . Such an equivalence would not be true for all local systems, but Lemma 4.11 explains that the twist by the antipodal map of  $S^1$  is innocuous when it comes to unipotent local systems.

Finally, we can view Corollary 4.13 as a natural consequence of this interplay between the product  $[0, 1/2] \times S^1$  and the Möbius strip  $Moeb$ . Namely, we have seen that the left hand side can be naturally interpreted as twisted unipotent local systems on  $Moeb$ , while the right hand side can be naturally interpreted as twisted unipotent local systems on  $[0, 1/2] \times S^1$ .

**4.4. Application to Langlands parameters.** Let us focus on the most interesting case of trivial monodromicity when  $\alpha$  is the identity  $e \in H$ . By Corollary 4.13, we have a canonical identification

$$\mathcal{L}_e/G \simeq \coprod_{\iota \in I} \mathcal{L}^u(K_\iota \backslash G/B).$$

where the monodromic Langlands parameter variety  $\mathcal{L}_e$  consists of the data

$$\mathcal{L}_e = \{(g, B) \in G \times \mathcal{B} \mid g\eta(g) \in G^u \cap B\},$$

where  $G^u \subset G$  is the formal neighborhood of the unipotent elements.

Now our results from [BN2], as recalled in Section 2, immediately provide a canonical equivalence of dg categories

$$\mathrm{QCoh}(\mathcal{L}_e/G)_{loc}^{\mathbb{S}^-} \simeq \oplus_{\iota \in I} \mathcal{D}(K_\iota \backslash G/B).$$

Observe that the  $\infty$ -category  $\oplus_{\iota \in I} \mathcal{D}(K_\iota \backslash G/B)$  forms the categorical Langlands parameters for trivial infinitesimal character.

Thus we arrive at the following fundamental interpretation of Langlands parameters.

**Corollary 4.15.** *The localized  $\mathbb{S}$ -invariants with bounded above weight in the dg category  $\mathrm{QCoh}(\mathcal{L}_e/G)$  are canonically equivalent to the categorical Langlands parameters for trivial infinitesimal character.*

For general monodromicity  $\alpha \in H$ , again by Corollary 4.13, we have a canonical identification

$$\mathcal{L}_\alpha/G \simeq \coprod_{\iota \in I} \mathcal{L}^u(K(\alpha)_\iota \backslash G(\alpha)/B(\alpha)).$$

Thus we can transport the canonical  $\mathbb{S}$ -action on the unipotent loop space of the right hand side to the monodromic Langlands parameter variety of the left hand side. Again our results from [BN2] immediately provide a canonical equivalence of  $\infty$ -categories

$$\mathrm{QCoh}(\mathcal{L}_\alpha/G)_{loc}^{\mathbb{S}^-} \simeq \oplus_{\iota \in I} \mathcal{D}(K(\alpha)_\iota \backslash G(\alpha)/B(\alpha)).$$

Thus we arrive at the following interpretation of Langlands parameters for arbitrary regular infinitesimal character.

**Corollary 4.16.** *For any regular  $\alpha \in H$ , the localized rotational  $\mathbb{S}$ -invariants with weight bounded above in the dg category  $\mathrm{QCoh}(\mathcal{L}_\alpha/G)$  are canonically equivalent to the categorical Langlands parameters at infinitesimal character  $W \cdot \alpha$ .*

The above result gives a description of  $\mathcal{D}$ -modules on the geometric parameter spaces

$$\coprod_{\iota \in I} K(\alpha)_\iota \backslash G(\alpha) / B(\alpha)$$

as part of a nice family with respect to the parameter  $\alpha$ . Namely, they can be recovered from quasicoherent sheaves on the unipotent loop spaces of the geometric parameter spaces. And the unipotent loop spaces in turn fit into the nice family formed by the Langlands parameter variety  $\mathcal{L}$ . It is crucial that we sought such a uniform picture in the realm of loop spaces rather than cotangent bundles.

*Remark 4.17.* By Lemma 4.10, we have a canonical identification

$$\mathcal{L}_\alpha / G \simeq (St_{\alpha, \eta(\alpha)} / G)^{\Gamma_{tw}}.$$

equipping  $\mathcal{L}_\alpha / G$  with a canonical  $S^1$ -action. As we discussed in Remark 3.8 in the setting of Steinberg varieties, one should proceed carefully with respect to this symmetry. In general, this  $S^1$ -action on  $\mathcal{L}_\alpha / G$  and the  $BG_a$ -action induced from its realization as a unipotent loop space do not exactly coincide. Rather the action map of one can be obtained from that of the other by the universal automorphism given by the central loop  $\alpha$ .

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